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Motion in a Plane

1 DECODING THE MOTION IN A PLANE

1.1 Scalars, Vectors and Tensors

- **Scalars** are physical quantities that have magnitude but no directional orientation such as density and temperature .
- **Vectors** are physical quantities with magnitude and a single direction such as velocity and force and obeys the triangle law of addition or equivalently the parallelogram law of addition.
- **Tensors** are physical quantities that have magnitude and two or more directions such as stress and strain. Tensors have invariant physical properties that are coordinate independent.

1.2 SOME PROPERTIES OF VECTORS

- **Equality of Two Vectors**
  Two vectors A and B may be defined to be equal if they have the same magnitude and point in the same direction.
  For many purposes, two vectors A and B may be defined to be equal if they have the same magnitude and point in the same direction. That is, A = B only if A = B and if A and B point in the same direction along parallel lines. For example, all the vectors in Figure 3.5 are equal even though they have different starting points. This property allows us to move a vector to a position parallel to itself in a diagram without affecting the vector.

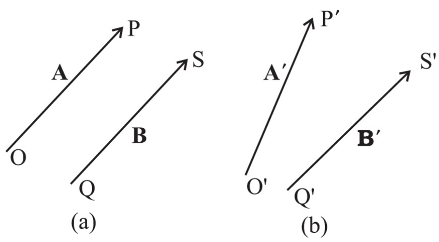

Figure 1.1: (a) Two equal vectors A and B. (b) Two vectors $A'$and A' are unequal though they are of the same length.

- **Negative of a Vector**
  The negative of the vector A is defined as the vector that when added to A gives zero for the vector sum. That is, A + (-A) = 0. The vectors A and -A have the same magnitude but point in opposite directions.

- **Multiplying a Vector by a Scalar**
  Multiplying a vector A with a positive number λ gives a vector whose magnitude is changed by the factor λ but the direction is the same as that of A :

- **ADDITION AND SUBTRACTION OF VECTORS — GRAPHICAL METHOD**
  Let us consider two vectors A and B that lie in a plane.

  The lengths of the line segments representing these vectors are proportional to the magnitude of the vectors.

  To find the sum $A+B$, we place vector B so that its tail is at the head of the vector A. Then, we join the tail of A to the head of B. This line OQ represents a vector R, that is, the sum of the vectors A and B.

  this graphical method is called the head-to-tail method. The two vectors and their resultant form three sides of a triangle, so this method is also known as triangle method of vector addition.

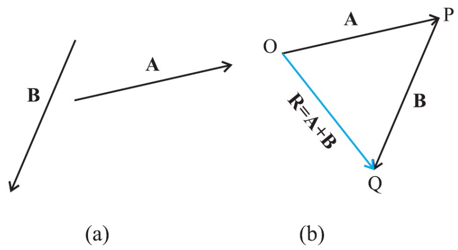

Figure 1.2: (a) Vectors A and B. (b) Vectors A and B added graphically.

2 COMPONENTS OF A VECTOR AND UNIT VECTORS

2.1 Unit vectors:

A unit vector is a vector of unit magnitude and points in a particular direction.

- It has no dimension and unit and is used to specify a direction only.

- Unit vectors along the x-, y and z-axes of a rectangular coordinate system are denoted by $\hat{i}, \hat{j}, \hat{k}$, respectively.
- These unit vectors are perpendicular to each other and the magnitudes are:

$$|\hat{i}| = |\hat{j}| = |\hat{k}| = 1 \tag{2.1}$$

In general, a vector A can be written as

$$A = |A|\hat{n} \tag{2.2}$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} \tag{2.3}$$

where $\hat{n}$ is a unit vector along A.

2.2 Resolution of vectors (In 2-D):

Let a vector $\vec{A}$ is making an angle θ with X axis. It can be resolved into two components along X axis and Y axis.
In triangle

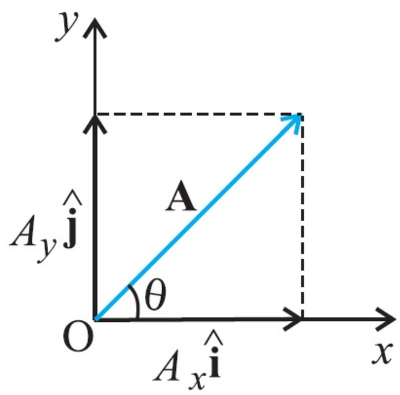

Figure 2.1:

$$\cos\theta = \frac{A_x}{A} \tag{2.4}$$

$$A_x = A\cos\theta \tag{2.5}$$

$$\sin\theta = \frac{A_y}{A} \tag{2.6}$$

$$A_y = A\sin\theta \tag{2.7}$$

Squaring and adding equations (2.5) and (2.7)

$$A_x^2 + A_y^2 = A^2\cos^2\theta + A^2\sin^2\theta \tag{2.8}$$

$$A_x^2 + A_y^2 = A^2 \tag{2.9}$$

$$A = \sqrt{A_x^2 + A_y^2} \tag{2.10}$$

On dividing (2.5) and (2.7)

$$\frac{A\sin\theta}{A\cos\theta} = \frac{A_y}{A_x} \tag{2.11}$$

$$\tan\theta = \frac{A_y}{A_x} \tag{2.12}$$

2.3 Resolution of vectors (In 3-D):

To resolve a general vector A into three components along x-, y-, and z-axes in three dimensions. If α, β, γ are the angles between A and the x-, y-, and z-axes, respectively.

$$A_x = A\cos\alpha, A_y = A\cos\beta, A_z = A\cos\gamma \tag{2.13}$$

In general, a vector A can be written as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \tag{2.14}$$

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{2.15}$$

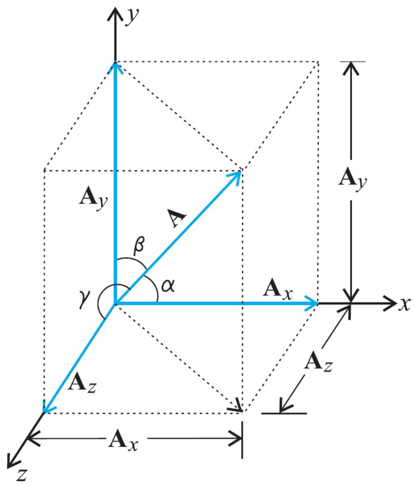

Figure 2.2: A vector A resolved into components along x-, y-, and z-axes

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \tag{2.16}$$

where x, y, and z are the components of $\vec{r}$ along x-, y-, z-axes, respectively.

2.4 VECTOR ADDITION – ANALYTICAL METHOD

Consider two vectors $\vec{A}$ and $\vec{B}$ in x-y plane with components A_x, A_y, A_z and B_x, B_y, B_z :

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \tag{2.17}$$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k} \tag{2.18}$$

Let $\vec{R}$ be their sum.

$$\vec{R} = \vec{A} + \vec{B} \tag{2.19}$$

$$\vec{R} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} + B_x\hat{i} + B_y\hat{j} + B_z\hat{k} \tag{2.20}$$

Arrange and regroup the vectors

$$\vec{R} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k} \tag{2.21}$$

$$\vec{R} = R_x\hat{i} + R_y\hat{j} + R_z\hat{k} \tag{2.22}$$

where,

$$R_x = (A_x + B_x), R_y = (A_y + B_y), R_z = (A_z + B_z) \tag{2.23}$$

3 PARALLELOGRAM OF VECTOR ADDITION

If two vectors are represented by the two adjacent sides of a parallelogram. Then the resultant of two vectors is given by the diagonal of the parallelogram.

3.1 Proof:

Let two vectors $\vec{A}$ and $\vec{B}$ are represented by two adjacent sides of a parallelogram. Making an angle θ. The resultant $\vec{R}$ is represented by a diagonal OS which is making an angle α with the $\vec{A}$.

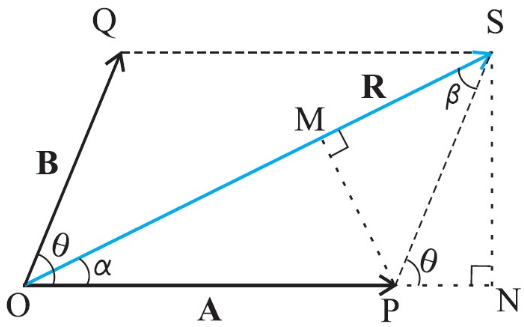

Figure 3.1:

From ΔSON

$$OS^2 = ON^2 + SN^2 \tag{3.1}$$

$$OS^2 = (OP + PN)^2 + SN^2 \tag{3.2}$$

Where,
SN=$B\sin\theta$ and PN=$B\cos\theta$

$$OS^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2 \tag{3.3}$$

$$R^2 = A^2 + B^2\cos^2\theta + 2AB\cos\theta + B^2\sin^2\theta \tag{3.4}$$

$$R^2 = A^2 + B^2 + 2AB\cos\theta \tag{3.5}$$

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta} \tag{3.6}$$

$$\tan\alpha = \frac{SN}{ON} \tag{3.7}$$

$$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta} \tag{3.8}$$

4 MOTION IN A PLANE

4.1 Position and Displacement Vectors

- **Position Vectors** are physical quantities that have magnitude but no directional orientation such as density and temperature .
- **Displacement Vectors** are physical quantities with magnitude and a single direction such as velocity and force and obeys the triangle law of addition or equivalently the parallelogram law of addition.
- **Tensors** are physical quantities that have magnitude and two or more directions such as stress and strain. Tensors have invariant physical properties that are coordinate independent.

4.2 Velocity

The velocity (instantaneous velocity) is given by the limiting value of the average velocity as the time interval approaches zero :
The direction of velocity at any point on the path of an object is tangential to the path at that point and is in the direction of motion.

4.3 Acceleration

The acceleration (instantaneous acceleration) is the limiting value of the average acceleration as the time interval approaches zero :

5 PROJECTILE MOTION

- An object that is in flight after being thrown or projected is called a **projectile**.
- Suppose an object is projected with initial velocity **u**, making an angle θ with x-axis.
- The velocity of a projectile have two components:

1. The horizontal component $u\cos\theta$, which remain constant throughout the motion
2. The vertical component $u\sin\theta$ which changes with time.

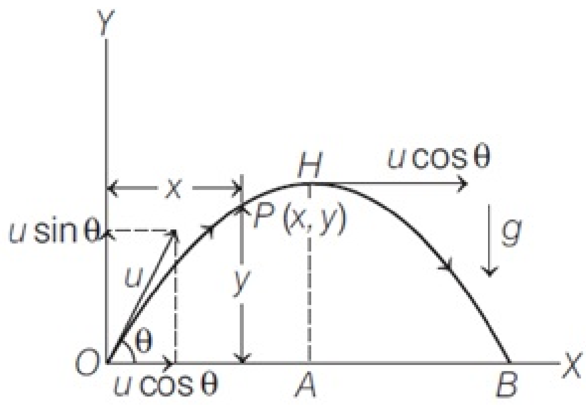

Figure 5.1: Motion of an object projected with velocity.

5.1 Equation of Trajectory

Let object reaches a point P(x,y) after time t.
Horizontal Distance, $x = HorizontalVelocity \times time$

$x = u\cos\theta . t$

$$\implies t = \frac{x}{u\cos\theta} \tag{5.1}$$

For vertical motion:
$u_y = u\sin\theta, a = -g$
The vertical distance covered in time t is :

$$y = u_y t + \frac{1}{2}at^2 \tag{5.2}$$

$$y = u\sin\theta t - \frac{1}{2}gt^2 \tag{5.3}$$

substituting the value of t from equation 5.1

$$y = u\sin\theta . \frac{x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2 \tag{5.4}$$

$$y = x.\tan\theta - \frac{g}{2u^2\cos^2\theta}.x^2 \tag{5.5}$$

The trajectory of a projectile is a Parabola.

5.2 Time of maximum height

Let t_m be the time taken by the projectile to reach the maximum height H.
At highest point, $v_y = 0$
as $v_y = u_y + at$

$$0 = u\sin\theta - g t_m \tag{5.6}$$

$$t_m = \frac{u\sin\theta}{g} \tag{5.7}$$

Time of Flight (T_f)

$$\boxed{T_f = 2.t_m} \tag{5.8}$$

$$T_f = \frac{2u\sin\theta}{g} \tag{5.9}$$

5.3 Maximum height

At highest point, $v_y = 0$.
As

$$v_y^2 - u_y^2 = 2aH \tag{5.10}$$

$$0^2 - u^2\sin^2\theta = 2(-g)H \tag{5.11}$$

$$H = \frac{u^2\sin^2\theta}{2g} \tag{5.12}$$

5.4 Horizontal Range (R)

$$R = u\cos\theta \times T_f \tag{5.13}$$

$$R = \frac{u^2\sin 2\theta}{g} \tag{5.14}$$

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